Propositional Logic for Epistemic Logic

July 28, 2014

Propositional Logic

A proposition ((indicative) sentence, (closed) wff), in the context of propositional logic, is the kind of linguistic entity that can have a truth value.

Propositional Logic

A proposition ((indicative) sentence, (closed) wff), in the context of propositional logic, is the kind of linguistic entity that can have a truth value.

Propositional logic studies the logical properties of complex propositions (entailment, logical truth, consistency etc.) based on their propositional parts.

The Language of PL

Symbols:

- Propositional letters: $p, q, r, s\ldots$
- Connectives:
  - Unary: $\neg$
  - Binary: $\wedge, \vee, \Rightarrow, \Leftrightarrow$
- Punctuation: (,)

The Language of PL

Symbols:

- Propositional letters: $p, q, r, s...$
- Connectives:
  - Unary: $\neg$
  - Binary: $\wedge, \vee, \Rightarrow, \Leftrightarrow$
- Punctuation: (,)

Well-formed formulas (wff):

1) Every propositional letter is a wff.
2) If ϕ is a wff, then $(\neg\phi)$ is a wff.
3) If ϕ and ψ are wffs, so are $(\phi \wedge \psi)$, $(\phi \vee \psi)$, $(\phi \Rightarrow \psi)$ and $(\phi \Leftrightarrow \psi)$.
4) Only items received by 1-3 are wffs.

Semantics

The connectives of PL denote *truth functions*: the truth value of a wff ϕ is a function of the truth values of the sub-formulas of ϕ.

Semantics

The connectives of PL denote *truth functions*: the truth value of a wff ϕ is a function of the truth values of the sub-formulas of ϕ.

A valuation v is a function from the set of propositional letters to $\{T, F\}$.

Semantics

The connectives of PL denote *truth functions*: the truth value of a wff ϕ is a function of the truth values of the sub-formulas of ϕ.

A valuation v is a function from the set of propositional letters to $\{T, F\}$. We can view v as a *model*, an *interpretation* or as representing a *state of affairs*.

Semantics

The connectives of PL denote *truth functions*: the truth value of a wff ϕ is a function of the truth values of the sub-formulas of ϕ.

A valuation v is a function from the set of propositional letters to $\{T, F\}$. We can view v as a *model*, an *interpretation* or as representing a *state of affairs*.

We extend v to a *satisfaction relation* on all wff's inductively, in the following way:

Semantics

The connectives of PL denote *truth functions*: the truth value of a wff ϕ is a function of the truth values of the sub-formulas of ϕ.

A valuation v is a function from the set of propositional letters to $\{T, F\}$. We can view v as a *model*, an *interpretation* or as representing a *state of affairs*.

We extend v to a *satisfaction relation* on all wff's inductively, in the following way:

$$\text{For a propositional letter } p, \quad \models_v p \text{ iff } v(p) = T$$

Semantics

The connectives of PL denote *truth functions*: the truth value of a wff ϕ is a function of the truth values of the sub-formulas of ϕ.

A valuation v is a function from the set of propositional letters to $\{T, F\}$. We can view v as a *model*, an *interpretation* or as representing a *state of affairs*.

We extend v to a *satisfaction relation* on all wff's inductively, in the following way:

$$\text{For a propositional letter } p, \quad \models_v p \text{ iff } v(p) = T$$
$$\models_v \neg\phi \text{ iff } \not\models_v \phi$$

Semantics

The connectives of PL denote *truth functions*: the truth value of a wff ϕ is a function of the truth values of the sub-formulas of ϕ.

A valuation v is a function from the set of propositional letters to $\{T, F\}$. We can view v as a *model*, an *interpretation* or as representing a *state of affairs*.

We extend v to a *satisfaction relation* on all wff's inductively, in the following way:

$$\text{For a propositional letter } p, \quad \models_v p \text{ iff } v(p) = T$$
$$\models_v \neg\phi \text{ iff } \not\models_v \phi$$
$$\models_v \phi \wedge \psi \text{ iff } \models_v \phi \text{ and } \models_v \psi$$

Semantics

The connectives of PL denote *truth functions*: the truth value of a wff ϕ is a function of the truth values of the sub-formulas of ϕ.

A valuation v is a function from the set of propositional letters to $\{T, F\}$. We can view v as a *model*, an *interpretation* or as representing a *state of affairs*.

We extend v to a *satisfaction relation* on all wff's inductively, in the following way:

$$\text{For a propositional letter } p, \quad \models_v p \text{ iff } v(p) = T$$
$$\models_v \neg\phi \text{ iff } \not\models_v \phi$$
$$\models_v \phi \wedge \psi \text{ iff } \models_v \phi \text{ and } \models_v \psi$$
$$\models_v \phi \vee \psi \text{ iff } \models_v \phi \text{ or } \models_v \psi$$

Semantics

The connectives of PL denote *truth functions*: the truth value of a wff ϕ is a function of the truth values of the sub-formulas of ϕ.

A valuation v is a function from the set of propositional letters to $\{T, F\}$. We can view v as a *model*, an *interpretation* or as representing a *state of affairs*.

We extend v to a *satisfaction relation* on all wff's inductively, in the following way:

$$\text{For a propositional letter } p, \quad \models_v p \text{ iff } v(p) = T$$
$$\models_v \neg\phi \text{ iff } \not\models_v \phi$$
$$\models_v \phi \wedge \psi \text{ iff } \models_v \phi \text{ and } \models_v \psi$$
$$\models_v \phi \vee \psi \text{ iff } \models_v \phi \text{ or } \models_v \psi$$
$$\models_v \phi \Rightarrow \psi \text{ iff } \models_v \psi \text{ or } \not\models_v \phi$$

Semantics

The connectives of PL denote *truth functions*: the truth value of a wff ϕ is a function of the truth values of the sub-formulas of ϕ.

A valuation v is a function from the set of propositional letters to $\{T, F\}$. We can view v as a *model*, an *interpretation* or as representing a *state of affairs*.

We extend v to a *satisfaction relation* on all wff's inductively, in the following way:

$$\text{For a propositional letter } p, \quad \models_v p \text{ iff } v(p) = T$$
$$\models_v \neg\phi \text{ iff } \not\models_v \phi$$
$$\models_v \phi \wedge \psi \text{ iff } \models_v \phi \text{ and } \models_v \psi$$
$$\models_v \phi \vee \psi \text{ iff } \models_v \phi \text{ or } \models_v \psi$$
$$\models_v \phi \Rightarrow \psi \text{ iff } \models_v \psi \text{ or } \not\models_v \phi$$
$$\models_v \phi \Leftrightarrow \psi \text{ iff } \models_v \phi \text{ and } \models_v \psi \text{ or } \not\models_v \phi \text{ and } \not\models_v \psi$$

Semantics

The set of connectives $\{\neg, \wedge, \vee, \Rightarrow, \Leftrightarrow\}$ is *functionally complete*: for all n, any boolean function $f : \{T, F\}^n \rightarrow \{T, F\}$ can be represented by a wff in PL.

Semantics

The set of connectives $\{\neg, \wedge, \vee, \Rightarrow, \Leftrightarrow\}$ is *functionally complete*: for all n, any boolean function $f : \{T, F\}^n \rightarrow \{T, F\}$ can be represented by a wff in PL.

The following sets of connectives are functionally complete:

$\{\neg, \vee, \wedge\}$

$\{\neg, \vee\}$

$\{\neg, \wedge\}$

$\{\neg, \Rightarrow\}$

Semantics

The set of connectives $\{\neg, \wedge, \vee, \Rightarrow, \Leftrightarrow\}$ is *functionally complete*: for all n, any boolean function $f : \{T, F\}^n \rightarrow \{T, F\}$ can be represented by a wff in PL.

The following sets of connectives are functionally complete:

$\{\neg, \vee, \wedge\}$

$\{\neg, \vee\}$

$\{\neg, \wedge\}$

$\{\neg, \Rightarrow\}$

Ex.: Show that $\{\vee, \Rightarrow\}$ is *not* functionally complete.

Semantics

Let Γ be a set of wff's, and let ϕ be a wff.
Γ *entails* ϕ ($\Gamma \models \phi$) if for every valuation function v, if $\models_v \psi$ for every $\psi \in \Gamma$, then $\models_v \phi$.

We shall say that a set Γ of wffs is *(semantically) consistent* if there is a valuation v such that $\models_v \psi$ for every $\psi \in \Gamma$.

Proof theory

The system H

Axioms (schemas):

P1. $\phi \Rightarrow (\psi \Rightarrow \phi)$

P2. $(\phi \Rightarrow (\psi \Rightarrow \xi)) \Rightarrow ((\phi \Rightarrow \psi) \Rightarrow (\phi \Rightarrow \xi))$

P3. $(\neg\phi \Rightarrow \neg\psi) \Rightarrow (\psi \Rightarrow \phi)$

Rule of inference:

(MP)

$$\frac{\phi \quad \phi \Rightarrow \psi}{\psi}$$

Proof theory

A *derivation* (=*proof*) of a sentence ϕ from a set of sentences Γ is a finite sequence of sentences such that ϕ is the last sentence in the sequence, and for each sentence ψ in the sequence,

Proof theory

A *derivation* (=*proof*) of a sentence ϕ from a set of sentences Γ is a finite sequence of sentences such that ϕ is the last sentence in the sequence, and for each sentence ψ in the sequence,

1) $\psi \in \Gamma$,

Proof theory

A *derivation* (=*proof*) of a sentence ϕ from a set of sentences Γ is a finite sequence of sentences such that ϕ is the last sentence in the sequence, and for each sentence ψ in the sequence,

1) $\psi \in \Gamma$,

or

2) ψ is an axiom,

Proof theory

A *derivation* (=*proof*) of a sentence ϕ from a set of sentences Γ is a finite sequence of sentences such that ϕ is the last sentence in the sequence, and for each sentence ψ in the sequence,

1) $\psi \in \Gamma$,

or

2) ψ is an axiom,

or

3) ψ results from applying an inference rule to previous sentences in the sequence.

Proof theory

A *derivation* (=*proof*) of a sentence ϕ from a set of sentences Γ is a finite sequence of sentences such that ϕ is the last sentence in the sequence, and for each sentence ψ in the sequence,

1) $\psi \in \Gamma$,

or

2) ψ is an axiom,

or

3) ψ results from applying an inference rule to previous sentences in the sequence.

A sentence ϕ is *provable* from a set of sentences Γ ($\Gamma \vdash_H \phi$) if there is a derivation of ϕ from Γ.

Proof theory

A *derivation* (=*proof*) of a sentence ϕ from a set of sentences Γ is a finite sequence of sentences such that ϕ is the last sentence in the sequence, and for each sentence ψ in the sequence,

1) $\psi \in \Gamma$,

or

2) ψ is an axiom,

or

3) ψ results from applying an inference rule to previous sentences in the sequence.

A sentence ϕ is *provable* from a set of sentences Γ ($\Gamma \vdash_H \phi$) if there is a derivation of ϕ from Γ. A sentence ϕ is a *theorem* if ϕ is provable from the empty set ($\vdash_H \phi$).

Proof theory

A *derivation* (=*proof*) of a sentence ϕ from a set of sentences Γ is a finite sequence of sentences such that ϕ is the last sentence in the sequence, and for each sentence ψ in the sequence,

1) $\psi \in \Gamma$,

or

2) ψ is an axiom,

or

3) ψ results from applying an inference rule to previous sentences in the sequence.

A sentence ϕ is *provable* from a set of sentences Γ ($\Gamma \vdash \phi$) if there is a derivation of ϕ from Γ. A sentence ϕ is a *theorem* if ϕ is provable from the empty set ($\vdash \phi$).

Soundness

If ϕ is provable from Γ, then ϕ is entailed by Γ.

Soundness

If ϕ is provable from Γ, then ϕ is entailed by Γ.

$$\Gamma \vdash \phi \rightarrow \Gamma \models \phi$$

Soundness

If ϕ is provable from Γ in H, then ϕ is entailed by Γ in PL.

$$\Gamma \vdash_H \phi \rightarrow \Gamma \models_{PL} \phi$$

Completeness

If ϕ is entailed by Γ, then ϕ is provable from Γ.

Completeness

If ϕ is entailed by Γ, then ϕ is provable from Γ.

$$\Gamma \models \phi \rightarrow \Gamma \vdash \phi$$

Completeness

If ϕ is entailed by Γ in PL, then ϕ is provable from Γ in H.

$$\Gamma \models_{PL} \phi \rightarrow \Gamma \vdash_{H} \phi$$

- ‘It is necessary that φ’
- ‘It is possible that φ’

- 'It is necessary that φ'
- 'It is possible that φ'
- $\Box\varphi$
- $\Diamond\varphi$

- ‘It is necessary that φ’
- ‘It is possible that φ’
- $\Box\varphi$
- $\Diamond\varphi$
- If φ is a wff, so is $\Box\varphi$ ($\Diamond\varphi$)

Axioms

- (T) $\Box\varphi \Rightarrow \varphi$

Axioms

- (T) $\Box\varphi \Rightarrow \varphi$
- (4) $\Box\varphi \Rightarrow \Box\Box\varphi$

Axioms

- (T) $\Box\varphi \Rightarrow \varphi$
- (4) $\Box\varphi \Rightarrow \Box\Box\varphi$
- (5) $\Diamond\varphi \Rightarrow \Box\Diamond\varphi$

Axioms

- (T) $\Box\varphi \Rightarrow \varphi$
- (4) $\Box\varphi \Rightarrow \Box\Box\varphi$
- (5) $\Diamond\varphi \Rightarrow \Box\Diamond\varphi$
- (B) $\varphi \Rightarrow \Box\Diamond\varphi$

Axioms

- (T) $\Box\varphi \Rightarrow \varphi$
- (4) $\Box\varphi \Rightarrow \Box\Box\varphi$
- (5) $\Diamond\varphi \Rightarrow \Box\Diamond\varphi$
- (B) $\varphi \Rightarrow \Box\Diamond\varphi$
- (D) $\Box\varphi \Rightarrow \Diamond\varphi$

Axioms

- (N) $\vdash \varphi \ / \vdash \Box\varphi$

Axioms

- (N) $\vdash \varphi \; / \vdash \Box\varphi$
- (K) $\Box(\varphi \Rightarrow \psi) \Rightarrow (\Box\varphi \Rightarrow \Box\psi)$

- KT = PL + N + K + T
- KD4 = PL + N + K + D + 4

- KT = PL + N + K + T
- KD4 = PL + N + K + D + 4
- KT4 (S4)
- KT5 (S5)

Semantics

A	$\neg A$
w	f
f	w

A	$\Box A$
w	?
f	f

A	$\Diamond A$
w	w
f	?

Interlude: relations

- 'x bears relation R to y'
- xRy or $x \rightarrow y$

Interlude: relations

- 'x bears relation R to y'
- xRy or $x \to y$
- (Reflexivity) for all x: xRx
- (Symmetry) for all x, y: $xRy \Rightarrow yRx$
- (Transitivity) for all x, y, z: $(xRy \wedge yRz) \Rightarrow xRz$
- (Euclidian) for all x, y, z: $(xRy \wedge xRz) \Rightarrow yRz$
- (Seriality) for all x, there exists y: xRy

Interlude: relations

- Relations that are reflexive, symmetric and transitive are called *equivalence relations*.

Interlude: relations

- Relations that are reflexive, symmetric and transitive are called *equivalence relations*.
- Relations that are reflexive and transitive are called *preorders*.

- Let W be a non-empty set of worlds.
- $v : Atoms \times W \rightarrow \{T, F\}$

- Let W be a non-empty set of worlds.
- $v : Atoms \times W \rightarrow \{T, F\}$
- As before we extend v into a *satisfaction relation* on all wff's inductively
- $w \models_v p = v(p, w)$

- Let W be a non-empty set of worlds.
- $v : Atoms \times W \rightarrow \{T, F\}$
- As before we extend v into a *satisfaction relation* on all wff's inductively
- $w \models_v p = v(p, w)$
- $w \models_v \neg\varphi$ iff $w \not\models_v \varphi$
- $\ldots$

- Let W be a non-empty set of worlds.
- $v : Atoms \times W \to \{T, F\}$
- As before we extend v into a *satisfaction relation* on all wff's inductively
- $w \models_v p = v(p, w)$
- $w \models_v \neg\varphi$ iff $w \not\models_v \varphi$
- $\ldots$
- $w \models_v \Box\varphi$ iff for all w' such that wRw' : $w' \models_v \varphi$

- Let W be a non-empty set of worlds.
- $v : Atoms \times W \to \{T, F\}$
- As before we extend v into a *satisfaction relation* on all wff's inductively
- $w \models_v p = v(p, w)$
- $w \models_v \neg\varphi$ iff $w \not\models_v \varphi$
- ...
- $w \models_v \Box\varphi$ iff for all w' such that wRw' : $w' \models_v \varphi$
- $w \models_v \Diamond\varphi$ iff for some w' such that wRw' : $w' \models_v \varphi$

- A frame $\mathcal{F}$ is an ordered pair (W, R)
- A model $\mathcal{M}$ is a triple (W, R, v)

- A frame $\mathcal{F}$ is an ordered pair (W, R)
- A model $\mathcal{M}$ is a triple (W, R, v)
- $\mathcal{M}$ is said to be ‘based’ on $\mathcal{F}$

- A wff φ is valid on a frame (W, R) iff

- A wff φ is valid on a frame (W, R) iff
- for every model (W, R, v) based on (W, R)

- A wff φ is valid on a frame (W, R) iff
- for every model (W, R, v) based on (W, R)
- and every $w \in W$,

- A wff φ is valid on a frame (W, R) iff
- for every model (W, R, v) based on (W, R)
- and every $w \in W$,
- $w \models_v \varphi$

Axioms

- (N) $\vdash \varphi \;/\; \vdash \Box\varphi$
- (K) $\Box(\varphi \Rightarrow \psi) \Rightarrow (\Box\varphi \Rightarrow \Box\psi)$

- (T) $\Box\varphi \Rightarrow \varphi$
- (4) $\Box\varphi \Rightarrow \Box\Box\varphi$
- (5) $\Diamond\varphi \Rightarrow \Box\Diamond\varphi$
- (B) $\varphi \Rightarrow \Box\Diamond\varphi$
- (D) $\Box\varphi \Rightarrow \Diamond\varphi$

Countermodel to 4

- Let $\mathcal{M} = (W, R, v)$

Countermodel to 4

- Let $\mathcal{M} = (W, R, v)$
- $W = \{w, w', w''\}$
- $R = \{(w, w'), (w', w'')\}$
- $v(p, w) = v(p, w') = T$
- $v(p, w'') = F$

- (T) $\Box\varphi \Rightarrow \varphi$ reflexive
- (4) $\Box\varphi \Rightarrow \Box\Box\varphi$ transitive
- (5) $\Diamond\varphi \Rightarrow \Box\Diamond\varphi$ Euclidian
- (B) $\varphi \Rightarrow \Box\Diamond\varphi$ symmetric
- (D) $\Box\varphi \Rightarrow \Diamond\varphi$ serial