Exercises - Solutions

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1. Give examples of the following types of relations: We may regard relations extensionally for our purposes. That is, a binary relation R on a set X is a subset of the Cartesian product of X: $R \subseteq X \times X$, where the $X \times X$ is just the set of all ordered pairs that can be formed by means of the members of X ($X \times X = \{(y, z);$for $y \in X$ and $z \in X\}$). Hence, the following 'gerrymandered' relations on the set $X = \{a, b, c\}$ are perfectly acceptable:

   - symmetric $\{(a, b), (b, a)\}$
   - serial $\{(a, b), (b, c), (c, a)\}$
   - non-transitive $\{(a, b), (b, c), (c, a)\}$ because we have aRb and bRc but not aRc.
   - Euclidian $\{(a, b), (a, c), (b, c), (c, b)\}$

2. Find a model that validates:

   - T and 4, but not B and 5
     - $W = \{w, w', w''\}$
     - $R = \{(w, w'), (w, w''), (w'', w'')\}$
     - $w \models_v p$
     - $w \models_v \Diamond p$
     - $w \models_v \neg\Box p$
     - $w' \models_v p$
     - $w' \models_v \neg\Diamond p$
     - $w'' \models_v \neg p$
   - D, 4, 5, but not T and B
     - $W = \{w, w', w,'', w'''\}$
     - $R = \{(w, w'), (w'', w''')\}$
     - $w \models_v \neg p$
     - $w \models_v \Diamond p$
     - $w \models_v \Box p$
     - $w' \models_v p$
     - $w' \models_v \Diamond p$

    - $w' \models_v \Box p$
    - $w'' \models_v p$
    - $w''' \models_v \neg p$
    - $w''' \models_v \neg \Diamond p$
- B and 4, but not D
    - $W = \{w\}$
    - $R = \emptyset$
    - $w \models_v \neg p$
    - $w \models_v \Box p$
    - $w \models_v \neg \Diamond p$
- D and 5, but not 4
    - $W = \{w, w', w,''\}$
    - $R = \{(w, w'), (w', w'), (w', w''), (w'', w')\}$
    - $w \models_v \Box p$
    - $w \models_v \Diamond p$
    - $w' \models_v p$
    - $w' \models_v \neg \Box p$
    - $w' \models_v \Diamond p$
    - $w'' \models_v \neg p$
- B and D, but not T
    - $W = \{w, w'\}$
    - $R = \{(w, w'), (w', w')\}$
    - $w \models_v \neg p$
    - $w \models_v \Box p$
    - $w \models_v \Diamond p$
    - $w' \models_v p$
    - $w' \models_v \Diamond p$

3. Determine whether

- KB4 = KT5 Given the soundness and completeness results we stated, it is enough to establish that all and only symmetric and transitive relations are reflexive and Euclidian. But this is not so. Counterexample: Let $R = \emptyset$.
- KB4 = KB5 We need to establish that all and only symmetric and transitive relations are Euclidian. Proof of the left-to-right direction: Let R be a symmetric and transitive relation. Suppose wRw' and wRw''. By symmetry, $w'Rw$. Hence, by transitivity, $w'Rw''$ as required.
  Proof of the right-to-left direction: Let R be a symmetric and Euclidian relation. Suppose wRw' and $w'Rw''$. By symmetry, $w'Rw$. By transitivity, $w'Rw''$ as required.