Solutions for Exercises for Propositional Logic

1. Proof by induction.

2.
   - A tautology: $(p \vee (\neg p \vee (q \vee (r \vee s))))$
   - A contradiction: $(p \wedge (\neg p \wedge (q \wedge (r \wedge s))))$
   - A contingent sentence: $(p \wedge (q \wedge (r \wedge s)))$

3. (a) In order to show functional completeness of $\{\neg, \vee\}$ we need to show that for every formula ϕ whose connectives are included in $\{\neg, \wedge, \vee, \Rightarrow, \Leftrightarrow\}$ is equivalent to a formula ψ whose connectives are included in $\{\neg, \vee\}$.

   Proof by induction.

   - Base case: $\phi = p$ for some propositional letter p - trivial.
   - Inductive step:
     - Assume $\phi = (\neg\chi)$ and that χ is equivalent to a formula ψ_0 whose connectives are included in $\{\neg, \vee\}$. Then $\psi = \neg\psi_0$ is equivalent to ϕ and its connectives are included in $\{\neg, \vee\}$.
     - Assume $\phi = (\chi_0 \vee \chi_1)$: similar to the negation case.
     - Assume $\phi = (\chi_0 \wedge \chi_1)$ and that χ_0 and χ_1 are equivalent to ψ_0 and ψ_1 respectively, the latter's connectives are included in $\{\neg, \vee\}$. Define $\psi = (\neg(\neg\psi_0 \vee \neg\psi_1))$.
     - Assume $\phi = (\chi_0 \Rightarrow \chi_1)$ and that χ_0 and χ_1 are equivalent to ψ_0 and ψ_1 respectively, the latter's connectives are included in $\{\neg, \vee\}$. Define $\psi = (\neg\psi_0 \vee \psi_1)$.
     - Assume $\phi = (\chi_0 \Leftrightarrow \chi_1)$ and that χ_0 and χ_1 are equivalent to ψ_0 and ψ_1 respectively, the latter's connectives are included in $\{\neg, \vee\}$. Define $\psi = (\neg(\neg(\neg\psi_0 \vee \psi_1) \vee \neg(\neg\psi_1 \vee \psi_0)))$.

   (b) In order to show that $\{\vee, \Rightarrow\}$ is not functionally complete one needs to show that there is a formula ϕ in PL that is not equivalent to any formula using only $\vee$ and $\Rightarrow$.

   Outline of proof: Let $\phi = \neg p$. The main claim follows from:

   Claim: Let v be the valuation that assigns T to all propositional letters. For every formula ψ, if ψ uses only $\vee$ and $\Rightarrow$, then ψ is satisfied by v.

   Proof: by induction.

4. (a) Every consistent set of wffs has an inconsistent subset.—False. The set $\{p\}$ is consistent, and both its subsets, $\{p\}$ and $\emptyset$ are consistent.

   (b) Every inconsistent set of wffs has a consistent subset.—True. The empty set is a subset of every set and is consistent. However, not every inconsistent set of wffs has a non-empty consistent subset—consider $\{p \wedge \neg p\}$.

(c) If $\{\phi, \psi\}$ is consistent, then $\{\phi, \neg\psi\}$ is inconsistent.—False. Consider $\{p, q\}$ and $\{p, \neg q\}$.

(d) If $\{\phi, \psi\}$ is inconsistent, then $\phi \models \neg\psi$.—True. If $\{\phi, \psi\}$ is inconsistent, then there is no valuation v that satisfies ϕ and ψ, so every valuation v that satisfies ϕ does not satisfy ψ and thus satisfies $\neg\psi$, and so $\phi \models \neg\psi$.

5. (a) $\vdash_H ((p \Rightarrow \neg q) \Rightarrow s) \Rightarrow (((r \Rightarrow \neg q) \Rightarrow s) \Rightarrow ((p \Rightarrow \neg q) \Rightarrow s))$ — instance of axiom P2.

(b) $\vdash_H (p \Rightarrow p)$ — By the definition of satisfaction for $\Rightarrow$, $\models_v (p \Rightarrow p)$ for every v, and thus $\models (p \Rightarrow p)$, and then by completeness we have $\vdash (p \Rightarrow p)$.

(c) $\neg p \nvdash_H \neg q$ — Let v be the valuation such that $v(p) = F$ and $v(q) = T$. Then v satisfies $\neg p$ and does not satisfy $\neg q$, so $\neg p \not\models \neg q$. Thus by contraposing soundness, $\neg p \nvdash_H \neg q$.

(d) $\nvdash_H p \Rightarrow (q \Rightarrow \neg q)$ — Similar to (c), using the valuation v such that $v(p) = v(q) = T$.