Lecture 6: Arrow's Theorem and Judgment Aggregation

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We often need collective decisions representing compromises between individuals.

- Given the facts about what is good for each of us, what is good for the group as a whole?
- Which policy best balances the disparate preferences of citizens in a democracy?
- What is the best way arrive at a consensus based on the opinions of different experts?

Trouble With Majority Rule

Condorcet's Paradox Three friends, A, B, and C, are deciding where to go for lunch: Veggie Palace (V), the all-you-can-eat buffet at Indian Kitchen (I), or Steak House (S). A is a vegan, and cares mostly about the number of vegetarian options on the menu; B is American, and wants a place that does decent fries; C is just ravenous, and wants a hearty meal. Their preferences are as follows:

	A	B	C
first preference	V	S	I
second preference	I	V	S
third preference	S	I	V

- The majority prefer V to I.
- The majority prefer I to S.
- The majority prefer S to V.

The Doctrinal Paradox Gertrude trips and falls on the stairs in her building, and sues her landlady Deb, claiming that the stairs were in poor condition. The case is evaluated by three judges, who must come to a consensus on whether Deb is liable to pay damages. They vote on three propositions.

p	The stairs were in poor condition.
o	Deb was contractually obligated keep the stairs in good condition.
$l = p \wedge o$	Deb is liable for the stairs being in poor condition; i.e., the stairs were in poor condition and Deb was contractually obligated to keep them in good condition.

The judges (named A, B, and C) vote as follows

	p	o	$c = p \wedge o$
A	Yes	Yes	Yes
B	No	Yes	No
C	Yes	No	No
Majority	Yes	Yes	No

Is this just a problem with majority rule? How bad are things?

Judgment

Vocabulary

A Group N is a set of individuals of fixed size n.

A Logic consists of a set of expressible propositions $\mathcal{L}$, together with an entailment relation $\vDash$ that relates each set of sentences A to an individual sentence p.

$\mathcal{L}$ is closed under negation: (if $p \in \mathcal{L}$, then $\neg p \in \mathcal{L}$).

A set of sentences $A \subseteq \mathcal{L}$ is **inconsistent** iff for some $p \in \mathcal{L}$, $A \vDash p$ and $A \vDash \neg p$.

A set sentences $A \subseteq \mathcal{L}$ is **minimal inconsistent** iff it is inconsistent, but none of its subsets are inconsistent.

Self-Entailment For all $p \in \mathcal{L}$, $p \vDash p$.

Monotonicity For all $p \in \mathcal{L}$ and $A \subseteq B \subseteq \mathcal{L}$, if $A \vDash p$, then $B \vDash p$

Completability For all $A \subseteq \mathcal{L}$ and $p \in \mathcal{L}$, if A is consistent, then either $A \cup \{p\}$ is consistent or $A \cup \{\neg p\}$ is consistent.

An Agenda is a subset of $\mathcal{L}$–representing the propositions that we want the group to come to a consensus on.

The agenda is closed under negation: if it contains p, then it also contains $\neg p$.

Judgment Sets are subsets of $\mathcal{X}$, judged to be true by some entity or other. Individuals in the group have **individual judgment sets**. We'd like to assign a **collective judgment set** to the entire group, representing a consensus opinion.

A judgment set A is:

consistent iff it is a consistent set as defined above, i.e., there is no $p \in \mathcal{L}$ such that $A \vDash p$ and $A \vDash \neg p$.

complete iff for every $p \in \mathcal{X}$, either $A \vDash p$, or $A \vDash \neg p$.

A Profile $(A_1, A_2, \ldots A_n)$ is a sequence of judgment sets (where A_i represents the judgment set of the ith individual).

An Aggregation Rule F is a function mapping each admissible profile of individual judgment sets to a collective judgment set. ("Admissible" because some profiles might not be in its domain—e.g., the inconsistent ones.) Call F's domain $\mathcal{D}(F)$.

Examples:

Majority Rule For each $(A_1, A_2, \ldots A_n) \in \mathcal{D}(F)$, if $|\{A_i : p \in A_i\}| > n/2$, then $p \in F(A_1, A_2, \ldots A_n)$; else $p \notin F(A_1, A_2, \ldots A_n)$.

(There's a problem about how to interpret majority rule when n is even; one way to get around this is only to consider odd-sized n.)

Dictatorship For some particular person x in the group, for each $(A_1, A_2, \ldots A_n) \in \mathcal{D}(F)$, $F(A_1, A_2, \ldots A_n) = A_x$.

Inverse Dictatorship For some particular person x in the group, for each $(A_1, A_2, \ldots A_n) \in \mathcal{D}(F)$, $p \in F(A_1, A_2, \ldots A_n)$ iff $p \notin A_x$.

Properties of Agendas

Path-Connectedness For any $p, q \in \mathcal{X}$, we write $p \vDash^* q$ iff there is some $Y \subseteq \mathcal{X}$ which is consistent with p, consistent with q, and inconsistent with $\{p, \neg q\}$. An agenda is **path-connected** iff, for every contingent $p, q \in \mathcal{X}$, there exist $p_1, p_2, \ldots p_k \in \mathcal{X}$, with $p = p_1$ and $q = p_k$, such that $p_1 \vDash^* p_2 \vDash^* \ldots \vDash^* p_k$.

Non-Simplicity An agenda $\mathcal{X}$ is **non-simple** iff it has at least one minimal inconsistent subset $Y \subset \mathcal{X}$ with $|Y| \geq 3$.

Constraints on Judgment Aggregation Rules

Universal Domain The domain of F is the set of profiles of consistent, complete judgment sets.

Collective Rationality The range of F is the set of consistent, complete judgment sets.

Non-Dictatorship There is no individual x such that for all profiles $(A_1, A_2, \ldots A_n) \in \mathcal{D}(F)$, $F(A_1, A_2, \ldots A_n) = A_x$.

Independence For any proposition $p \in \mathcal{X}$
and profiles $(A_1, A_2, \ldots A_n), (A_1^*, A_2^*, \ldots A_n^*) \in \mathcal{D}(F)$,

if for every $i \in N$

$p \in (A_1, A_2, \ldots A_n)$ iff $p \in (A_1^*, A_2^*, \ldots A_n^*)$,

then $p \in F(A_1, A_2, \ldots A_n)$ iff $p \in F(A_1^*, A_2^*, \ldots A_n^*)$.

Unanimity For any profile $(A_1, A_2, \ldots A_n) \in \mathcal{D}(F)$ and any proposition $p \in \mathcal{X}$,

if for every $i \in N$,

$p \in A_i$

then $p \in F(A_1, A_2, \ldots A_n)$.

A Version of Arrow's Theorem

For path-connected, non-simple agendas, there is no judgment aggregation rule that satisfies Universal Domain, Collective Rationality, Non-Dictatorship, Independence, and Unanimity.

Contagion, or Field Expansion, Lemma

Almost Decisiveness $M \subseteq N$ is **almost decisive over** $p \in \mathcal{X}$ iff, for all profiles $(A_1, A_2, \ldots A_n) \in \mathcal{D}(F)$ such that

for all $i \in M$, $p \in A_i$, and

for all $i \notin M$, $p \notin A_i$

$p \in F(A_1, A_2, \ldots A_n)$ M is **almost decisive** iff M is almost decisive over all members of $\mathcal{X}$.

Decisiveness $M \subseteq N$ is **decisive over** $p \in \mathcal{X}$ iff, for all profiles $(A_1, A_2, \ldots A_n) \in \mathcal{D}(F)$ such that

iff for all $i \in M$, $p \in A_i$

$p \in F(A_1, A_2, \ldots A_n)$

M is decisive iff M is decisive over all members of $\mathcal{X}$.

Versions of this lemma say:

1. Every coalition that is almost-decisive over one proposition p is almost-decisive.
2. Every coalition that is decisive over one proposition p is decisive.
3. Every coalition that is almost-decisive over one proposition p is decisive.

I'lll prove version 3, using Universal Domain, Unanimity, Group Consistency, and Path-Connectedness.

Suppose that some (possibly empty) $Q = \{q_1, q_2 \ldots q_n\}$ is consistent with both p and r, and that $\{p\} \cup Q \vDash r$. Suppose that M is almost decisive over p. Then $\mathcal{D}(F)$ must contain some profiles with that fit the following rubric (where A, B, or both may be empty) :

	p	Q	$\neg r$
$i \in M$	Yes	Yes	No
$j \in A \subseteq N \backslash M$	No	Yes	Yes
$k \in B = (N \backslash M) \backslash A$	No	Yes	No

- By Unanimity, $Q \subseteq F(A_1, A_2, \ldots A_n)$.
- If M is almost decisive over p, then Collective Rationality requires that in any profile fitting this rubric, $r \in F(A_1, A_2, \ldots A_n)$.
- So by Independence, in any profile where $p \in A_i$ for every $in \in M$, $r \in F(A_1, A_2, \ldots A_n)$, regardless of what is happening in the middle column.
- So M is almost-decisive over r.
- Given Path-Connectedness, we can get from any p to any r in steps of this nature.

Group-Contraction Lemma

Let S be among the smallest almost-decisive groups. (We know there is one, because N is decisive and finite.) We can show that S has only one member, using Universal Domain, Unanimity, Collective Rationality, and Independence.

Let j be some particular individual in S, and suppose again that $Q = \{q_1, q_2 \ldots q_n\}$ is consistent with both p and $\neg r$, and that $\{p\} \cup Q \cup \{\neg r\}$ is minimal inconsistent. (Since we need Q to be non-empty, we need to assume that the agenda is non-simple.) Then by Universal Domain, $\mathcal{D}(F)$ must contain some profiles with the following features (where either $S \backslash \{j\}$ or $N \backslash S$ may be empty):

	p	Q	$\neg r$
j	Yes	Yes	No
$i \neq j \in S$	No	Yes	Yes
$k \in N \backslash S$	Yes	No	Yes

- By our assumption that S is a *smallest* almost decisive group, we know that if $S \setminus j$ is non-empty, it is not decisive, so for any profile $(A_1, A_2, \ldots A_n)$ fitting the rubric, $p \in F(A_1, A_2, \ldots A_n)$.
- By Unanimity, for any profile $(A_1, A_2, \ldots A_n)$ fitting the rubric, $Q \subseteq F(A_1, A_2, \ldots A_n)$.
- By Collective Rationality, in any profile $(A_1, A_2, \ldots A_n)$ fitting the collective, $r \subseteq F(A_1, A_2, \ldots A_n)$. So j is almost decisive for r.
- By the Contagion Lemma, j (a single individual) is decisive.