Lecture 5: Risk

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Problems for Expected Utility Maximizers

Pizza

My brother and I both prefer one pizza to a 50/50 gamble between 2 pizzas and no pizzas. I get satiated. My brother has an insatiable appetite for pizza, but is risk-averse; he doesn't want to take the chance of going hungry.

Hats

I like pizza, I like silly hats, and neither affects my enjoyment of the other. You will flip a fair coin, and give me a silly hat if it lands heads. You then offer me a choice: either you will give me a pizza as a consolation prize if I don't get the hat, or you will flip the coin twice, and give me a pizza if it lands heads the second time. Being risk averse, I prefer the consolation prize. In other words, I prefer Lottery 1 to Lottery 2.

	HH	**HT**	**TH**	**TT**
Lottery 1	pizza	pizza	hat	hat
Lottery 2	pizza + hat	pizza	hat	nothing

Ellsberg Paradox

The Sure-Thing Principle For all acts f, g, x, and y, and events E,

$$f_E x \succ g_E x \text{ iff } f_E y \succ g_E y$$

Expected utility theory entails the Sure-Thing Principle. In this example and the next, violating the Sure-Thing Principle looks rational.

An urn contains 30 red balls, and 60 balls that are either blue or yellow. A single ball is drawn. What are your preferences among these lotteries?

Bet Red $100 if a red ball is drawn; nothing otherwise

Bet Blue $100 if a blue ball is drawn; nothing otherwise

Bet Red or Yellow $100 if a red ball or a yellow ball is drawn; nothing otherwise

Bet Blue or Yellow $100 if a blue ball or a yellow ball is drawn; nothing otherwise

Many people prefer Lottery 1 to Lottery 2, and Lottery 4 to Lottery 3. This contradicts the **Sure-Thing Principle** and expected utility theory!

	Red	**Blue**	**Yellow**
Bet Red	$100	$0	$0
Bet Blue	$0	$100	$0
Bet Red or Yellow	$100	$0	$100
Bet Blue or Yellow	$0	$100	$100

Allais Paradox

What are your preferences between the following pairs of lotteries?

L_1	$1,000,000	with probability 1
L_2	$1,000,000 $5,000,000 $0	with probability 0.89 with probability 0.1 with probability 0.01
L_3	$1,000,000 $0	with probability 0.11 with probability 0.89
L_4	$5,000,000 $0	with probability 0.1 with probability 0.9

Many people prefer 1 to 2, and 4 to 3. This contradicts the **Sure-Thing Principle** and expected utility theory!

	1	**2-11**	**12-100**
L_1	$1,000,000	$1,000,000	$1,000,000
L_2	$0	$5,000,000	$1,000,000
L_3	$1,000,000	$1,000,000	$0
L_4	$0	$5,000,000	$0

Possible Explanations

- Diminishing marginal utility of goods?
  - Implausible in **Pizza**—there's a distinction between me and my brother.
  - Implausible in **Hats**—filling in other aspects of my psychology ought to establish that there is no interaction between different goods, without making my preferences unintelligible or irrational.
  - Unworkable in the Ellsberg and Allais cases: there are no utilities compatible with the stated preferences.
  - Explaining risk-averse behaviour in terms of the diminishing marginal utility of money yields strange results. E.g., if I always turn down a 50/50 bet between losing $10 and gaining $11 whatever my initial wealth level, then there is no amount of money for which I am willing to bet $100 in a 50/50 gamble (Rabin).
- Risk is globally valuable or disvaluable: a gamble's value is the sum of its expected utility and some global value.
  - This is consistent with the data, but not explanatory: *why* should global properties of the gamble be valuable or disvaluable?
- Outcomes have path-dependent utilities: their values depend on how they are acquired. (Alternatively: outcomes haven't been correctly individuated in the examples.)
  - Other examples where this strategy works: Pettit's polite guest; Sen's horrified guest; Diamond's conscientious prize-giver.
  - **Hats** refigured:

	HH	**HT**	**TH**	**TT**
Lottery 1	pizza	pizza	hat	hat
Lottery 2	pizza + hat	pizza	hat	**regret**

or

	HH	**HT**	**TH**	**TT**
Lottery 1	pizza + **surety**	pizza + **surety**	hat + **surety**	hat + **surety**
Lottery 2	pizza + hat	pizza	hat	nothing

  - We can describe the cases in a way that makes these rewritings implausible—we can specify that the agent cares about global properties of the gamble.
  - Without constraints on which properties can influence rational preference among outcomes, this strategy threatens to become trivial.
- An extra factor, risk, interacts with probability and utility to determine the overall value of a gamble.

Risk-Weighted Expected Utility

Three factors determine the desirability of an act:

- A probability function P
- A utility function u
- An increasing risk function $r : [0,1] \to [0,1]$

For a gamble g with 2 possible outcomes, where o_1 is the worse outcome,

$$U(g) = u(o_1) + P(o_2)u(o_2)$$

$$REU(g) = u(o_1) + r(P(o_2))u(o_2)$$

In a gamble g with n possible outcomes, let o_1 be the worst outcome, let o_2 be the second-worst outcome, and let o_i be the ith-worst outcome. Let E_i be the event that A yields an outcome at least as good as o_i. Then

$$U(g) = \sum_{i=1}^{n} P(E_i)(u(o_i) - u(o_{i-1}))$$

($u(o_0) = 0$, by convention.)

$$REU(g) = \sum_{i=1}^{n} r(P(E_i))(u(o_i) - u(o_{i-1}))$$

How does this handle the examples?

Pizza My brother and I can be distinguished by our preferences. Say that for me

$$r_R(p) = P$$

$$u_R(q) = \left(\frac{121}{64}\right)^{-x} - 1$$

For my brother Dan,

$$r_D(p) = \frac{p^2 + p}{2}$$

$$u_D(q) = q$$

Both of us are indifferent between a 50/50 gamble between 2 pizzas and 0 pizzas vs. 3/8 of a a pizza. But we have different attitudes toward a 50/50 gamble between 4 pizzas and 0 pizzas. I will pay about $\sqrt{11}/2\sqrt{2}$ pizzas for this gamble, which works out to scarcely more than 3/8 of a pizza. My brother will pay 3/4 of a pizza.

Hats I value hats and pizza equally. Since they are independent goods for me, Lottery 1 is a sure x utils for me, while Lottery 2 offers me a 25% chance of nothing, a 50% chance of x utils, and a 25% chance of $2x$ utils. If my utility function is concave, so that $r(0.5) < 0.5$ and $r(0.25) < 0.25$, I must think Lottery 2 is worse than Lottery 1.

Ellsberg No good answer.

Allais This works fine with many utility functions and concave risk functions. Suppose I value \$0 at 0 utiles, \$1,000,000 at 10 utiles, and \$5,000,000 at 20 utiles. And suppose $r(p) = p^2$. Then the utilities of lotteries are:

$REU(L_1) = 10 \qquad REU(L_2) = 9.901$
$REU(L_3) = 0.121 \qquad REU(L_4) = 2$

Features of REU

- Could also be computed top-down. Let $r^*(p) = 1 - r(1 - p)$. Then if we re-label the outcomes from best to worst, and let E_i^* be the event that g yields an outcome at most as good as o_i, we once again get

$$REU(g) = \sum_{i=1}^{n} r^*(P(E_i^*))(u(o_j) - u(o_{i-1}))$$

- Convex risk curves represent *risk-averse* preferences; concave risk curves represent *risk-seeking* preferences.
- Values of gambles are not affected by dividing up the state space. (All outcomes with the same value are run together.)
- We get only small effects by splitting an outcome into two similarly-valued outcomes with the same total probability.
- A crucial concept is that of a *comoncone*—a set of acts all of which (weakly) order the states in the same way, with respect to value. More formally, consider the following definitions.

  comonotonic: Acts a and b are comonotonic iff for all $s, s' \in \mathcal{S}$, $a(s) \succsim a(s')$ iff $b(s) \succsim b(s')$

  a comoncone is a set of acts, any two of which are comonotonic.

  Instead of the Sure-Thing Principle, we have

  The Comonotonic Sure-Thing Principle For all acts f, g, x and y, and events E,
  if $f_E x$, $g_E x$, $f_E y$ and $g_E y$ are comonotonic, then

$$f_E x \succ g_E x \text{ iff } f_E y \succ g_E y$$

Sure Losses?

Consider an agent with risk function $r(p) = p^2$, and a linear utility function for money.

The Dutch Book Suppose a fair coin is about to be flipped twice. What shall we say about the following bets?

	HH	**HT**	**TH**	**TT**	**REU**
Bet 1	\$200	\$100	\$100	\$0	62.50
Bet 2	\$100	\$100	\$0	\$0	25
Bet 3	\$100	\$0	\$100	\$0	25

The agent will pay \$62 for Bet 1, and sell each of bets 2 and 3 for \$26, right? But that results in a sure loss of \$10.

- Wrong!
- $REU(-\text{Bet 1}) \neq -REU(\text{Bet 1})$
- $REU(\text{Bet 1} + \text{Bet 2}) \neq REU(\text{Bet 1}) + REU(\text{Bet 2})$

The Mind-Changing Money Pump In a version of the Allais game, Rhoda is offered a choice between L_1 and a sweetened L_2+ (consisting of L_2 plus a guaranteed extra utile). A ticket numbered 1-100 is then drawn randomly. Rhoda learns whether the ticket is numbered 1-11, or 2-12. If Rhoda has chosen L_1, and the ticket is numbered 1-11, then Rhoda gets the opportunity to switch to an unsweetened L_2.

The Information-Avoiding Money Pump In the Allais game, a ticket numbered 1-100 is drawn randomly. Rhoda is then offered a choice between L_1 and L_2. Rhoda can decide to make the choice either before or after learning whether the ticket is numbered 1-11, or 2-12. If she chooses to learn the information, the deal will be sweetened with an extra utile.

These money pumps require a naive theory of choice.

Naive Choice At each node, choose the action that belongs to the best strategy (according to your current preferences) available at that node.

Sophisticated Choice Assume that if you reach a final choice node n, you will choose the action with the best outcome (according to your preferences at n). Assume that no other outcome is possible once you reach node n. Work backward through the tree until you reach the first choice node.

Resolute Choice Choose the best strategy at the first node (according to your preferences at the first node) and adhere to it at all other nodes, regardless of your later preferences.

References

Lara Buchak. *Risk and rationality*. Oxford University Press, 2013.