Day 2: Representation Theorems

Rachael Briggs

July 30, 2014

Some questions:

1. What is utility?
   - A functionalist proposal [Ramsey, 1931]: the higher the utility of an outcome, the stronger a person's preference for a lottery with a fixed chance of yielding that outcome.
2. Why think that people should (or do) maximize expected utility?
3. What is subjective probability?
   - A functionalist proposal: the higher the subjective probability, the shorter the odds at which one is willing to bet on that proposition.
4. Why think that people should (or do) assign subjective probabilities that obey the Kolmogorov axioms?

Representation theorems help answer the above questions by

- characterising the functional roles of probability and utility, and
- providing a key premise in an argument for maximising expected utility

Representation Argument for Maximising Expected Utility [Zynda, 2000]

The Rationality Condition The axioms of expected utility theory are the axioms of rational preference.

Representability If a person's preferences obey the axioms of expected utility theory, then he or she can be represented as having degrees of belief that obey the laws of the probability calculus [and a utility function such that she prefers acts with higher expected utility].

The Reality Condition If a person can be represented as having degrees of belief that obey the probability calculus [and a utility function such that she prefers acts with higher expected utility], then the person really has degrees of belief that obey the laws of the probability calculus [and really does prefer acts with higher expected utility].

$\therefore$ If a person [fails to prefer acts with higher expected utility], then that person violates at least one of the axioms of rational preference.

Preference and Representation

Preference is a two-place relation over the members of some domain. This domain varies from representation theorem to representation theorem.

- Lotteries [Von Neumann and Morgenstern, 1953]
  e.g., I'd prefer a 100% chance at a vegan curry over a 50-50 gamble between a lobster platter and going hungry.
- Acts [Savage, 1972]
  e.g., I'd rather crack the last egg into the same bowl as all the other eggs, than crack it into a new bowl
- Propositions [Jeffrey, 1983]
  e.g., I'd rather that it rain tomorrow than that I try to make a funny joke at dinner and fail

- The objects of preference are the objects of expected utility.
- I'll write

  $a \succ b$ for "a is strictly preferred to b"

  $a \sim b$ for "a is indifferent to b"

  $a \succsim b$ for "a is weakly preferred to b".
  (For our purposes, you can understand this as a disjunction: $a \succ b \vee a \sim b$.)

- I'll formulate everything in terms of weak preference $\succsim$. In all the preference orderings we'll be considering,

  $a \succ b$ iff $a \succsim b$ and $b \not\succsim a$

  $a \sim b$ iff $a \succsim b$ and $b \succsim a$

Representation (utilities) An expected utility function U (over a domain $\mathcal{D}$) represents a preference ordering $\succsim$ (over the same domain $\mathcal{D}$) if and only if

for all $a, b \in \mathcal{D}$,

$a \succsim b$ if and only if $U(a) \geq U(b)$

Von Neumann and Morgenstern's Representation Theorem

Vocabulary

$\mathcal{G}$ is a set of lotteries, or gambles: the objects of preference expected utility.

- It contains at least two **constant gambles**, which yield the same outcome no matter what.

- It is closed under a **mixing** operation:
  If $f, g \in \mathcal{G}$, and $0 \leq \alpha \leq 1$, then $\mathcal{G}$ contains
  $\alpha f \oplus (1-\alpha)g$
  (the gamble that yields f with probability α and g with probability $1-\alpha$).
- It contains only the constant gambles and gambles that can be constructed from them by this principle in a finite number of "mixing" steps.
- I will identify $\alpha f \oplus (1-\alpha)g$ with $(1-\alpha)g \oplus \alpha f$; and $1f \oplus 0g$ with f.

$\succsim$ is a weak preference relation over members of $\mathcal{G}$

Preference Axioms

Non-Triviality $\mathcal{G}$ contains at least two constant gambles f and g such that $f \succ g$.

Transitivity If $f \succsim g$, and $g \succsim h$, then $f \succsim h$.

Completeness For any $f, g \in \mathcal{G}$, either $f \succsim g$ or $g \succsim f$.

Averaging If $f \succ g$, then for every $\alpha \in (0, 1)$

$f \succ \alpha f \oplus (1-\alpha)g \succ g$

and if $f \succsim g$, then for every α,

$f \succsim \alpha f \oplus (1-\alpha)g \succsim g$

Independence $f \succ g$ iff $f \oplus h \succ g \oplus h$

and $f \succsim g$ iff $f \oplus h \succsim g \oplus h$

Continuity If $f \succ g \succ h$, then there is some α such that

$g \sim \alpha f \oplus (1-\alpha)h$

Reduction of Compound Lotteries $\alpha(\beta f \oplus (1-\beta)g) \oplus (1-\alpha)g \sim \alpha\beta f + (1-\alpha\beta)g$

The Theorem

For every space of gambles $\mathcal{G}$ and weak preference relation $\succsim$ satisfying the axioms, there is a function U with domain $\mathcal{G}$ such that

U **represents** $\succ$

$U(f) \geq U(g)$ iff $f \succsim g$

U **is an expected utility function**

$U(\alpha f \oplus (1-\alpha)g) = \alpha U(f) + (1-\alpha)U(g)$

and U **is unique up to positive linear transformation**

U' represents $\succ$ iff there exist real numbers $x > 0$ and y such that for every $g \in \mathcal{G}$

$xU(g) + y = U'(g)$

Sketch of a Proof

Assume a fixed domain of gambles $\mathcal{G}$ and preference relation $\succ$ satisfying the axioms. Show that there is an expected utility function, representing $\succ, \succsim$, and that U is unique up to positive linear transformation.

Definition of U

- Pick two gambles g_0 and g_1 such that $g_1 \succ g_0$.
- Let $U(g)$ be the γ such that:
  $$\begin{array}{ll} g \sim \gamma g_1 \oplus (1-\gamma) g_0 & \text{if } g_1 \succsim g \succsim g_0 \\ g_1 \sim \frac{1}{\gamma} g \oplus (1-\frac{1}{\gamma}) g_0 & \text{if } g \succ g_1 \\ g_0 \sim \frac{\gamma}{\gamma-1} g_1 \oplus \frac{-1}{\gamma-1} g & \text{if } g_0 \succ g \end{array}$$

Existence of γ (for a given g_0, g_1 and g) follows from the **Continuity Axiom**.

Uniqueness of γ (for a given g_0, g_1, and g) follows from **Reduction of Compound Lotteries**, **Averaging**, and **Transitivity**.

To show that γ is unique, we prove the

Better Chances Condition if $b \succ w$,
then $\alpha b \oplus (1-\alpha) w \succ \beta b \oplus (1-\beta) w$ iff $\alpha > \beta$

U represents $\succsim$

Consider (without loss of generality) an arbitrary f and g, such that $g_1 \succsim f \succsim g_0$ and $g_1 \succsim g \succsim g_0$.

We know that there exist exactly one α and exactly one β such that

$$f \sim \alpha g_1 \oplus (1-\alpha) g_0$$
$$g \sim \beta g_1 \oplus (1-\beta) g_0$$

By our definition of U,

$$U(f) > U(g) \text{ iff } \alpha > \beta$$

So by the **Better Chances Condition**,

$$U(f) > U(g) \text{ iff } f \succ g$$

U is an expected utility function

Again, consider (without loss of generality) an arbitrary f and g, such that $g_1 \succsim f \succsim g_0$ and $g_1 \succsim g \succsim g_0$.

We want to show that

for all γ such that $1 \geq \gamma \geq 0$,
$U(\alpha f \oplus (1-\alpha)g) = \alpha U(f) + (1-\alpha)U(g)$

1. We have proved that there exist exactly one α and exactly one β such that

   $f \sim \alpha g_1 \oplus (1-\alpha)g_0$
   $g \sim \beta g_1 \oplus (1-\beta)g_0$

2. By 1 and **Independence**,

   $\gamma f \oplus (1-\gamma)g \sim \gamma(\alpha g_1 \oplus (1-\alpha)g_0) \oplus (1-\gamma)(\beta g_1 \oplus (1-\beta)g_0)$

3. By 2 and **Reduction of Compound Lotteries**,

   $\gamma(\alpha g_1 \oplus (1-\alpha)g_0) \oplus (1-\gamma)(\beta g_1 \oplus (1-\beta)g_0)$
   $\sim \gamma\alpha + (1-\gamma)\beta)g_1 \oplus (1-(\gamma\alpha + (1-\gamma)\beta))g_0$

4. By 3 and the definition of U,

   $U(\gamma\alpha + (1-\gamma)\beta)g_1 \oplus (1-(\gamma\alpha + (1-\gamma)\beta))g_0) = \gamma\alpha + (1-\gamma)\beta$

5. Since U represents $\succsim$, 2 and 4 let us infer

   $U(\gamma f \oplus (1-\gamma)g) = \gamma\alpha + (1-\gamma)\beta$

6. By the definition of U,

   $U(f) = \alpha$
   $U(g) = \beta$

7. By 5 and 6, we have

   $U(\alpha f \oplus (1-\alpha)g) = \alpha U(f) + (1-\alpha)U(g)$

   as desired.

U is unique up to positive linear transformation (homework exercise)

(A Little About) Savage's Representation Theorem

- Savage adopts a subjective interpretation of probability. So he has to answer the question: where do the probabilities (used to build lotteries) come from?
- Answer: a representation theorem that builds probabilities and utilities from preferences.

Vocabulary

$\mathcal{O}$ is a set of outcomes

$\mathcal{S}$ is a set of states, and $\mathcal{F}$ is a set of events, or subsets of $\mathcal{S}$.

- Members of $\mathcal{F}$ are bearers of probabilities.

$\mathcal{A}$ is a set of acts, understood as finite-valued functions from states to outcomes.

$\succsim$ is a relation over acts

It can also relate sub-acts f_E and g_E, where f_E is the act f conditional on event E, and g_E is the act g conditional on the same event E.

- For each outcome, there is a constant act which yields that outcome in all states.
- If a and b are acts, and E is an event, then there is an act $a_E b$ which yields the same outcome as a in E states, and the same outcome as b in $\neg E$ states.

Highlights

- The axiom that receives the most discussion:

  The Sure-Thing Principle For all acts f and g, x, and y, and events E,

  $$f_E x \succ g_E x \text{ iff } f_E y \succ g_E y$$

  This says: your preference between two acts should depend only on what happens in the event that they yield different outcomes.

- Savage defines a subjective "weakly more probable than" relation $(\gtrsim)$ for events in terms of preference over acts.

  Definition of $(\gtrsim)$ Suppose a and b are constant acts such that $a \succ b$. Then
  $E(\gtrsim)F$ iff $a_E \& b_{\neg E} \succsim a_F \& b_{\neg F}$

- He constrains probability by imposing constraints on preference, which allows him to prove that $(\gtrsim)$ is represented by one (and only one) probability function.
- He defines gambles as equivalence classes of acts, and uses the von Neumann-Morgenstern theorem to prove that preferences satisfying his axioms can be represented by an expected utility function (together with the probability function that represents $(\gtrsim)$). Furthermore, the utility function is unique up to positive linear transformation.

References

Richard C Jeffrey. *The logic of decision*. 2nd edition, 1983.

Frank P Ramsey. Truth and Probability. In *The Foundations of Mathematics and other Logical Essays*, pages 1–41. 1931.

Leonard J Savage. *The Foundations of Statistics*. Courier Dover Publications, 1972.

J Von Neumann and Otto Morgenstern. *Theory of Games and Economic Behavior: 3d Ed*. 3rd edition, 1953.

L Zynda. Representation theorems and realism about degrees of belief. *Philosophy of Science*, 2000.