Summer School on Mathematical Philosophy for Female Students

Introduction to Probability Theory, Algebra, and Set Theory

Catrin Campbell-Moore and Sebastian Lutz*

July 28, 2014

Question 1. Draw Venn diagrams for the following sets and write them in a simpler way:

- $A \cap (B \cup A)$
- $A \setminus (B \cap (A \cup C))$

Question 2. Let $\Omega = \{\omega_1, \omega_2, \omega_3\}$ and $\mathcal{F} = \mathcal{P}(\Omega)$. Let $P(\{\omega_1\}) = 0.1$, $P(\{\omega_2\}) = 0.3$ and $P(\{\omega_3\}) = 0.6$. Find the probabilities for all the other events in $\mathcal{F}$.

Question 3. Two fair, independent, four-sided dice, one red and one green, are rolled. Let the event A be "The sum of the faces showing is an even number." Define an appropriate Ω, list the states in A and say what $P(A)$ is.

Question 4. Consider the following two events:

- A: It rains in Munich tomorrow
- B: The burglars who stole 300,000 litres of beer a few weeks ago will be caught

Assume $P(A) = 0.4$, $P(B) = 0.6$ and that A and B are probabilistically independent according to P.

Calculate:

1. $P(A^c)$
2. $P(A \cap B)$
3. $P(A \cup B)$

Question 5. Consider $\Omega = \{\omega_1, \omega_2\}$ and $\{\omega_1\}, \{\omega_2\} \in \mathcal{F}$. Let $P(\{\omega_1\}) = 0.2$ and $P(\{\omega_2\}) = 0.8$. Calculate the expected values of the following variables.

*Email catrin@ccampbell-moore.com with any mistakes or comments

	ω_1 $P(\{\omega_1\}) = 0.2$	ω_2 $P(\{\omega_2\}) = 0.8$
X	5	5
Y	6	8
Z	1	10

Question 6. For the roll of one four-sided and one six-sided die which are fair and independent,

1. what is the probability of the total score being 3?
2. what is the expectation value of the total score?
3. what is the expectation value of the maximum score?

Question 7. Assume that for the role of a die, $X(\{i \text{ on top}\}) = 2i$ and $Y(\{i \text{ on top}\}) = i^2$. Assume X is applied to the outcomes of a roll of an eight-sided die and Y is applied to the outcomes of a roll of a six-sided die. What is the probability of $X + Y = 6$?

Question 8. Suppose there is a medical diagnostic test for a disease. The sensitivity of the test is .95. This means that if a person has the disease, the probability that the test gives a positive response is .95. The specificity of the test is .90. This means that if a person does not have the disease, the probability that the test gives a negative response is .90, or that the false positive rate of the test is .10. In the population, 1% of the people have the disease. What is the probability that a person tested has the disease, given the results of the test is positive? Let D be the event that the person has the disease and T be the event that the test gives a positive result.

Question 9 (Monty Hall)**.** Suppose you are on a gameshow. There are three doors: D_A, D_B and D_C. Behind one of the doors there is an expensive sports car, behind the other two there are goats, but you don't know which. (You want the sports car!). Monty Hall, the gameshow host, asks you to pick one of the doors but not to open it yet. He will then pick one of the other two doors which has a goat behind it and show you the goat. He then gives you the opportunity to change your mind. Should you switch? Justify your answer using probability theory. Note: If you are standing in front of the door with the car, Monty will open one of the remaining doors at random.

Question 10. Show that a partition of Ω induces a partition on every element of $\mathcal{F}$. I.e. if $B_1, \dots, B_n$ partitions Ω then for every $A \in \mathcal{F}$, $(A \cap B_1), \dots, (A \cap B_n)$ partitions A.

Question 11. Show that the following follow from the formalism for events:

- If $A, B \in \mathcal{F}$ then $A \cap B \in \mathcal{F}$.
- If $A, B \in \mathcal{F}$ then $A \backslash B \in \mathcal{F}$.

Question 12. Prove that the following follow from the axioms of probability:

1. $P(\varnothing) = 0$
2. $P(A \cup B) + P(A \cap B) = P(A) + P(B)$

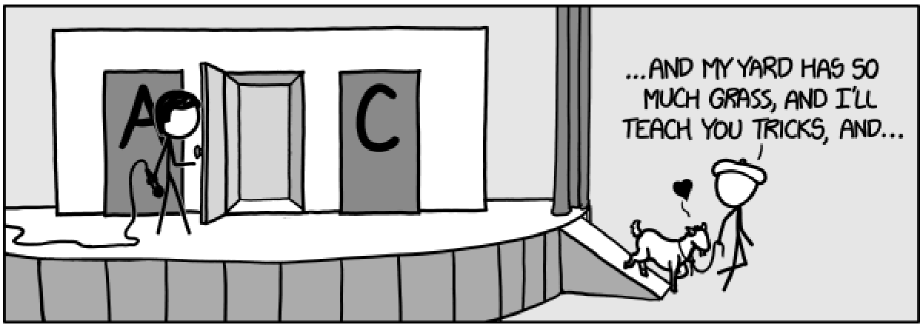

Figure 1: xkcd

3. If $C_1, \ldots, C_k$ is a partition of A then $P(A) = P(C_1) + \ldots + P(C_k)$[1]
4. The law of total probability.

Question 13. [2]Suppose that $\mathcal{F}$ is finite. Show that P satisfies the axioms of probability if and only if there is a $P^- : \{A | A \text{ is an atom of } \mathcal{F}\} \to \mathbb{R}$ such that

- $P^-(A) \geqslant 0$ for all atoms A
- $\sum_{A \text{ is an atom of } \mathcal{F}} P^-(A) = 1$

with

$$P(B) = \sum_{A \text{ is an atom and } A \subseteq B} P^-(A)$$

Hint: first show that if $B \in \mathcal{F}$ then

$$P(B) = \sum_{A \textit{ is an atom with } A \subseteq B} P(A)$$

Question 14. Prove that if A and B are independent then from knowing $P(A)$ and $P(B)$ one can find the probabilities of all the events in the Boolean algebra generated by A and B.

Question 15. Show that if A is probabilistically independent of B then B is probabilistically independent of A. I.e. if $P(A|B) = P(A)$ then $P(B|A) = P(B)$.

Question 16. Random variable X is an *indicator function* for $A \in \mathcal{F}$ if and only if

$$X(\omega) = \begin{cases} 1 & \text{if } \omega \in A \\ 0 & \text{if } \omega \notin A. \end{cases}$$

1. Show that the expected value of the indicator function of A is the probability of A.

[1]I have added this as an explicit part since I will use it as a lemma to prove the law of total probability, and also for question 13 so I want to be able to refer back to it.

[2]There was a part of this question which I'd forgotten to write up on the original problem sheet

2. Express the indicator functions for $A \cap B$, $A \cup B$ and A^c through the indicator functions for A and B.

Question 17. [3] Let Ω be infinite. Show that there can be no probability function defined on $\mathcal{F} = \mathcal{P}(\Omega)$ where for each $\omega, \omega' \in \Omega$, $P(\{\omega\}) = P(\{\omega'\}) > 0$.

Question 18 (The Birthday Problem)**.** There are n people in the room. Assume that peoples birthdays are equally likely to be on any day of the year. (And that for two different people, where their birthdays lie are independent). Ignore leap years.

What is the probability that at least one of them has the same birthday as you? How large does this need to be for the probability to be more than 0.5?

What is the probability that two people have their birthday on the same day? How large does this need to be for the probability to be more than 0.5?

[3] Note, this question has changed since the original problem sheet since the original question was just wrong.